

On $\left| \bar{N}, q_n; \delta_n \right|_k$ Summability Factors of Infinite Series (III)

Aradhana Dutt Jauhari

ABSTRACT: A theorem concerning some new absolute summability method is proved. Many other results some known and unknown are derived.

KEY WORDS AND PHRASES: Absolute Summability, Summability.



1- INTRODUCTION :

Let $\sum a_n$ be a given infinite series with partial sums S_n and $U_n = n a_n$. Let σ_n^α and t_n^α be the n^{th} Cesàro mean of order α ($\alpha > -1$) of the sequence S_n and U_n respectively. The series $\sum a_n$ is said to be summable $\left| C, \alpha \right|_k$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left| \sigma_n^\alpha - \sigma_{n-1}^\alpha \right|^k < \infty \quad (1.1)$$

Or equivalently,

$$\sum_{n=1}^{\infty} n^{-1} \left| t_n^\alpha \right|^k < \infty$$

where,

$$t_n^\alpha = n \sigma_n^\alpha - \sigma_{n-1}^\alpha$$

A series $\sum a_n$ is summable $\left| C, \alpha; \delta_n \right|_k$, if the series

$$\sum_{n=1}^{\infty} \delta_n n^k n^{k-1} \left| \sigma_n^\alpha - \sigma_{n-1}^\alpha \right|^k < \infty ; \quad (1.2)$$

$\delta(n)$, a positive non-decreasing sequence such that $\delta nm = \delta n \delta m$.

where σ_n^α is the n^{th} Cesàro mean of order α of $\sum a_n$. It follows that above definition reduces to that of FLETT [4].

Let p_n be a sequence of positive numbers such that,

$$P_n = \sum_{v=0}^n p_v \rightarrow \infty \text{ as } n \rightarrow \infty$$

The sequence -to -sequence transformation

$$T_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v \quad P_n \neq 0 (n \geq 0) \quad (1.3) \text{ defines the sequence of weighted means of the}$$

sequence S_n generated by the sequence of coefficients p_n . The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$ if

Email id- aditya_jauhari@rediffmail.com, Deptt. Of Applied Sciences (Mathematics), N.I.E.T., Greater Noida ,U.P. (INDIA)

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |u_n - u_{n-1}|^k < \infty \quad (1.4)$$

and it is said to be summable $|\bar{N}, p_n; \delta_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \delta \left(\frac{P_n}{p_n} \right)^k \left(\frac{P_n}{p_n} \right)^{k-1} |T_n - T_{n-1}|^k < \infty ; \quad (1.5)$$

In 1995, SULAIMAN, W.T. [8] proved the theorem. The main objective of this paper is to generalize the theorem of SULAIMAN[8]. However our theorem is as follows.

2-THEOREM:

Let $\{p_n\}$ and $\{q_n\}$ be two sequences of positive real numbers such that-

$$p_{n+1} = \square(p_n) \quad (2.1)$$

$$q_n = \square(q_{n+1}) \quad (2.2)$$

$$p_n Q_n = \square(P_n q_n) \quad (2.3)$$

The series $\sum a_n$ is summable $|\bar{N}, p_n, \delta_n|_k$ then $\sum a_n \varepsilon_n$ is summable $|\bar{N}, q_n, \delta_n|_k$ if -

$$\frac{Q_n}{q_n} \Delta \varepsilon_n = \square \left[\left(\frac{p_n Q_n}{P_n q_n} \right)^{1/k} \delta \left(\frac{P_n q_n}{p_n Q_n} \right) \right] \quad (2.4)$$

$$\varepsilon_n = \square \left[\left(\frac{p_n Q_n}{P_n q_n} \right)^{1/k} \delta \left(\frac{P_n q_n}{p_n Q_n} \right) \right] \quad (2.5)$$

3-PROOF OF THE THEOREM:

In order to prove the theorem it is sufficient to prove that by LEINDLER [5]

$$\sum_{n=1}^{\infty} \delta \left(\frac{P_n}{p_n} \right)^k \left(\frac{P_n}{p_n} \right)^{k-1} |T_n - T_{n-1}|^k < \infty$$

where ,

$$T_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v$$

Let

$$x_n = T_n - T_{n-1}, \quad n \geq 1, \quad s_0 = a_0$$

$$x_n = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^n P_{v-1} a_v$$

$$x_n \frac{P_n P_{n-1}}{p_n} = \sum_{v=1}^n P_{v-1} a_v$$

$$a_n = -\left(\frac{1}{P_{n-1}} \right) \Delta \left(x_{n-1} \frac{P_{n-1} P_{n-2}}{p_{n-1}} \right)$$

Considering-

$$\begin{aligned} t_n &= \frac{1}{Q_n} \sum_{v=0}^n q_v \sum_{\mu=0}^v a_\mu \epsilon_\mu \\ t_n &= \frac{1}{Q_n} \left[\sum_{v=0}^{n-1} Q_v (-a_{v+1} \epsilon_{v+1}) + Q_n \sum_{v=0}^n a_v \epsilon_v \right] \\ &= \frac{1}{Q_n} \left[-\sum_{v=0}^{n-1} Q_v (a_{v+1} \epsilon_{v+1}) \right] + \sum_{v=0}^n a_v \epsilon_v \end{aligned} \quad (3.1)$$

so,

$$\begin{aligned} t_n - t_{n-1} &= \left[-\frac{1}{Q_n} \sum_{v=0}^{n-1} Q_v (a_{v+1} \epsilon_{v+1}) + \right. \\ &\quad \left. \frac{1}{Q_{n-1}} \sum_{v=0}^{n-2} Q_v (a_{v+1} \epsilon_{v+1}) - \sum_{v=0}^{n-1} a_v \epsilon_v \right] + \sum_{v=0}^n a_v \epsilon_v \\ &= \left[-\frac{1}{Q_n} \sum_{v=0}^{n-1} Q_v (a_{v+1} \epsilon_{v+1}) + \frac{1}{Q_{n-1}} \sum_{v=0}^{n-2} Q_v (a_{v+1} \epsilon_{v+1}) \right] \\ &= \left[\frac{q_v}{Q_n Q_{n-1}} \sum_{v=0}^{n-1} Q_v (a_{v+1} \epsilon_{v+1}) \right] \\ &= \left[\frac{q_v}{Q_n Q_{n-1}} \sum_{v=0}^{n-1} Q_{v-1} a_v \epsilon_v \right] \\ &= \left[-\frac{q_v}{Q_n Q_{n-1}} \sum_{v=0}^n Q_{v-1} \epsilon_v \left(\frac{1}{P_{n-1}} \right) \Delta \left(x_{n-1} \frac{P_{n-1} P_{n-2}}{p_{n-1}} \right) \right] \end{aligned} \quad (3.2)$$

Using Abel's transformation, we have-

$$t_n - t_{n-1} = -\frac{q_n}{Q_n Q_{n-1}} \left[\sum_{v=0}^{n-1} \Delta \left(\frac{Q_{v-1} \epsilon_v}{P_{v-1}} \right) \sum_{\mu=0}^v \Delta \left(x_{\mu-1} \frac{P_{\mu-1} P_{\mu-2}}{p_{\mu-1}} \right) + \right]$$

$$\begin{aligned}
 & + \left(\frac{Q_{n-1} \varepsilon_n}{P_{n-1}} \right) \sum_{v=0}^n \Delta \left(x_{v-1} \frac{P_{v-1} P_{v-2}}{p_{v-1}} \right) \\
 = - \frac{q_n}{Q_n Q_{n-1}} & \left[\sum_{v=0}^{n-1} \Delta \left(\frac{Q_{v-1} \varepsilon_v}{P_{v-1}} \right) \left(-x_v \frac{P_v P_{v-1}}{p_v} \right) \right. \\
 & \left. + \left(\frac{Q_{n-1} \varepsilon_n}{P_{n-1}} \right) \left(-x_n \frac{P_n P_{n-1}}{p_n} \right) \right]
 \end{aligned}$$

Clearly,

$$\Delta \left(\frac{Q_{v-1} \varepsilon_v}{P_{v-1}} \right) = \frac{p_v}{P_v P_{v-1}} Q_{v-1} \varepsilon_v + \frac{Q_{v-1}}{P_v} \Delta \varepsilon_v - \frac{q_v \varepsilon_{v+1}}{P_v}$$

so,

$$\begin{aligned}
 t_n - t_{n-1} &= \left[\frac{q_n}{Q_n Q_{n-1}} \sum_{v=0}^{n-1} Q_{v-1} \varepsilon_v x_v + \frac{q_n}{Q_n Q_{n-1}} \sum_{v=0}^{n-1} x_v \left(\frac{P_{v-1}}{p_v} Q_{v-1} \Delta \varepsilon_v \right) \right. \\
 &\quad \left. - \left(\frac{q_n}{Q_n Q_{n-1}} \sum_{v=0}^{n-1} x_v \frac{P_{v-1}}{p_v} q_v \varepsilon_{v+1} \right) + \left(\frac{q_n}{Q_n} \frac{P_n}{p_n} x_n \varepsilon_n \right) \right] \\
 &= \sum_1 + \sum_2 + \sum_3 + \sum_4 \quad (\text{say})
 \end{aligned}$$

Now for $\sum a_n \varepsilon_n$ to be summable $\left| \bar{N}, q_n, \delta, n \right|_k$ if-

$$\sum_{n=1}^{\infty} \delta \left(\frac{Q_n}{q_n} \right)^k \left(\frac{Q_n}{q_n} \right)^{k-1} \left| T_n - T_{n-1} \right|^k < \infty \quad (3.3)$$

Now by Minkowski's inequality, we have-

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \delta \left(\frac{Q_n}{q_n} \right)^k \left(\frac{Q_n}{q_n} \right)^{k-1} \left| \sum_1 + \sum_2 + \sum_3 + \sum_4 \right|^k \\
 & \leq M \left[\sum_{n=1}^{\infty} \delta \left(\frac{Q_n}{q_n} \right)^k \left(\frac{Q_n}{q_n} \right)^{k-1} \left| \sum_1 \right|^k + \right. \\
 & \quad \left. + \sum_{n=1}^{\infty} \delta \left(\frac{Q_n}{q_n} \right)^k \left(\frac{Q_n}{q_n} \right)^{k-1} \left| \sum_2 \right|^k + \sum_{n=1}^{\infty} \delta \left(\frac{Q_n}{q_n} \right)^k \left(\frac{Q_n}{q_n} \right)^{k-1} \left| \sum_3 \right|^k + \right. \\
 & \quad \left. + \sum_{n=1}^{\infty} \delta \left(\frac{Q_n}{q_n} \right)^k \left(\frac{Q_n}{q_n} \right)^{k-1} \left| \sum_4 \right|^k \right]
 \end{aligned}$$

where M is some positive constant.

$$= \sum_{11} + \sum_{12} + \sum_{13} + \sum_{14} \quad (\text{say})$$

Now,

$$\begin{aligned}
 \sum_{11} &= M \sum_{v=1}^{\infty} \delta \left(\frac{Q_v}{q_v} \right)^k \left(\frac{Q_v}{q_v} \right)^{k-1} \left| \sum_1 \right|^k \\
 &= M \sum_{v=1}^{\infty} \delta \left(\frac{Q_v}{q_v} \right)^k \left(\frac{Q_v}{q_v} \right)^{k-1} \left| \frac{q_n}{Q_n Q_{n-1}} \sum_{v=0}^{n-1} Q_{v-1} \epsilon_v x_v \right|^k \\
 &= \boxed{\left[\sum_{n=1}^{\infty} \delta \left(\frac{Q_n}{q_n} \right)^k \left(\frac{Q_n q_n}{q_n} \right)^{k-1} \frac{1}{q_n^{k-1}} \frac{q_n^k}{Q_n^k Q_{n-1}^k} \left\{ \sum_{v=0}^{n-1} \frac{Q_{v-1} q_v}{q_v} |\epsilon_v| |x_v| \right\}^k \right]} \\
 &= \boxed{\left[\sum_{n=1}^{\infty} \delta \left(\frac{Q_n}{q_n} \right)^k \left(\frac{Q_n^{k-1}}{q_n^{k-1}} \right) \frac{q_n^k}{Q_n^k Q_{n-1}^k} \left\{ \sum_{v=0}^{n-1} \left(\frac{Q_{v-1}}{q_v} \right)^k |\epsilon_v|^k |x_v|^k q_v \right\} \left\{ \sum_{v=0}^{n-1} q_v \right\}^{n-1} \right]}
 \end{aligned}$$

Using Hölder's inequality-

$$\begin{aligned}
 &= \boxed{\left[\sum_{n=1}^{\infty} \delta \left(\frac{Q_n}{q_n} \right)^k \frac{q_n}{Q_n Q_{n-1}} \left\{ \sum_{v=0}^{n-1} \left(\frac{Q_v}{q_v} \right)^k \left(\frac{P_v}{p_v} \right)^{k-1} \left(\frac{p_v}{P_v} \right)^{k-1} |\epsilon_v|^k |x_v|^k q_v \right\} \right]} \\
 &= \boxed{\left[\sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \left\{ \sum_{v=0}^{n-1} \delta \left(\frac{Q_v}{q_v} \right)^k \left(\frac{P_v}{p_v} \right)^{k-1} \left(\frac{Q_v p_v}{q_v P_v} \right)^k \frac{P_v}{p_v} |\epsilon_v|^k |x_v|^k q_v \right\} \right]} \\
 &= \boxed{\left[\left\{ \sum_{v=1}^{n-1} \delta \left(\frac{Q_v}{q_v} \right)^k \left(\frac{P_v}{p_v} \right)^{k-1} \frac{P_v}{p_v} |\epsilon_v|^k |x_v|^k q_v \right\} \sum_{n=v+1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \right]} \quad [p_v Q_v = \boxed{P_v q_v}] \\
 &= \boxed{\left[\left\{ \sum_{v=0}^{n-1} \delta \left(\frac{Q_v}{q_v} \right)^k \left(\frac{P_v}{p_v} \right)^{k-1} \frac{P_v q_v}{Q_v p_v} |\epsilon_v|^k |x_v|^k \right\} \right]} \\
 &= \boxed{\left[\left\{ \sum_{v=0}^{n-1} \delta \left(\frac{Q_v}{q_v} \right)^k \left(\frac{P_v}{p_v} \right)^{k-1} \frac{P_v q_v}{Q_v p_v} |\epsilon_v|^k |x_v|^k \frac{Q_v p_v}{P_v q_v} \delta \left[\frac{P_v q_v}{Q_v p_v} \right]^k \right\} \right]} \\
 &= \boxed{\left[\left\{ \sum_{v=0}^{n-1} \delta \left(\frac{P_v}{p_v} \right)^k \left(\frac{P_v}{p_v} \right)^{k-1} |\epsilon_v|^k |x_v|^k \right\} \right]} \\
 &= \boxed{1}
 \end{aligned}$$

Again

$$\begin{aligned}
 \sum_{12} &= M \sum_{v=1}^{\infty} \delta \left(\frac{Q_v}{q_v} \right)^k \left(\frac{Q_v}{q_v} \right)^{k-1} \left| \sum_2 \right|^k \\
 &= \boxed{\left[\sum_{v=1}^{\infty} M \delta \left(\frac{Q_n}{q_n} \right)^k \left(\frac{Q_n}{q_n} \right)^{k-1} \left\{ \frac{q_n}{Q_n Q_{n-1}} \sum_{v=0}^{n-1} \left(\frac{P_{v-1}}{p_v} Q_{v-1} \Delta \epsilon_v x_v \right) \right\}^k \right]}
 \end{aligned}$$

$$\begin{aligned}
 &= \boxed{\left[\sum_{n=1}^{\infty} \delta \left(\frac{Q_n}{q_n} \right)^k \left(\frac{Q_n}{q_n} \right)^{k-1} \frac{q_n^k}{Q_n^k Q_{n-1}^k} \left\{ \sum_{v=0}^{n-1} \left(\frac{P_{v-1}}{p_v} Q_{v-1} |\Delta \epsilon_v| |x_v|^k \right) \right\}^k \right]} \\
 &= \boxed{\left\{ \sum_{n=1}^{\infty} \delta \left(\frac{Q_n}{q_n} \right)^k \frac{q_n}{Q_n Q_{n-1}^k} \left[\sum_{v=0}^{n-1} \left(\frac{P_{v-1}}{p_v} \frac{Q_{v-1} q_v}{q_v} |\Delta \epsilon_v| |x_v|^k \right) \right]^k \right\}} \\
 &= \boxed{\left\{ \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}^k} \left[\sum_{v=0}^{n-1} \left(\delta \left(\frac{Q_v}{q_v} \right)^k \left(\frac{P_{v-1}}{p_v} \right)^k \left(\frac{Q_{v-1}}{q_v} \right)^k q_v |\Delta \epsilon_v|^k |x_v|^k \right) \right] * \right.} \\
 &\quad \left. * \left\{ \sum_{v=0}^{n-1} q_v \right\}^{k-1} \right\}
 \end{aligned}$$

Using Hölder's inequality-

$$\begin{aligned}
 &= \boxed{\left\{ \sum_{n=1}^{\infty} \frac{q_n Q_{n-1}^{k-1}}{Q_n Q_{n-1}^k} \left[\sum_{v=0}^{n-1} \left(\delta \left(\frac{Q_v}{q_v} \right)^k \left(\frac{Q_v}{q_v} \right)^k \left(\frac{P_v}{p_v} \right)^k q_v |\Delta \epsilon_v|^k |x_v|^k \right) \right] \right\}} \\
 &= \boxed{\left\{ \sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \left[\sum_{v=0}^{n-1} \left(\left(\frac{P_v}{p_v} \right)^{k-1} |x_v|^k \delta \left(\frac{Q_v}{q_v} \right)^k \left(\frac{Q_v}{q_v} \right)^k \frac{P_v}{p_v} q_v |\Delta \epsilon_v|^k \right) \right] \right\}} \\
 &= \boxed{\left\{ \sum_{v=1}^{\infty} \left(\left(\frac{P_v}{p_v} \right)^{k-1} \frac{P_v}{p_v} |x_v|^k \delta \left(\frac{Q_v}{q_v} \right)^k \left(\frac{Q_v}{q_v} \right)^k |\Delta \epsilon_v|^k q_v \right) \sum_{n=v+1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \right\}} \\
 &= \boxed{\left\{ \sum_{v=1}^{\infty} \left(\left(\frac{P_v}{p_v} \right)^{k-1} \frac{P_v q_v}{p_v Q_v} |x_v|^k \delta \left(\frac{Q_v}{q_v} \right)^k \left(\frac{p_v Q_v}{q_v P_v} \right) \delta \left(\frac{P_v q_v}{p_v Q_v} \right)^k \right) \right\}} \\
 &\quad \text{(using condition (2.4) of theorem)} \\
 &= \boxed{\left\{ \sum_{v=1}^{\infty} \left(\delta \left(\frac{P_v}{p_v} \right)^k \left(\frac{P_v}{p_v} \right)^{k-1} |x_v|^k \right) \right\}} \\
 &= \boxed{(1)}
 \end{aligned}$$

Further,

$$\begin{aligned}
 \sum_{13} &= M \sum_{v=1}^{\infty} \delta \left(\frac{Q_v}{q_v} \right)^k \left(\frac{Q_v}{q_v} \right)^{k-1} \left| \sum_{v=0}^3 |x_v|^k \right| \\
 &= M \sum_{v=1}^{\infty} \delta \left(\frac{Q_v}{q_v} \right)^k \left(\frac{Q_v}{q_v} \right)^{k-1} \left| - \left(\frac{q_n}{Q_n Q_{n-1}} \sum_{v=0}^{n-1} x_v \frac{P_{v-1}}{p_v} q_v \epsilon_{v+1} \right) \right|^k \\
 &= M \sum_{v=1}^{\infty} \delta \left(\frac{Q_v}{q_v} \right)^k \left(\frac{Q_v}{q_v} \right)^{k-1} \left\{ \left(\frac{q_n^k}{Q_n^k Q_{n-1}^k} \sum_{v=0}^{n-1} |x_v| \frac{P_{v-1}}{p_v} q_v |\epsilon_{v+1}| \right)^k \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= \boxed{\sum_{n=1}^{\infty} \delta \left(\frac{Q_n}{q_n} \right)^k \left\{ \left(\frac{q_n}{Q_n Q_{n-1}} \sum_{v=0}^{n-1} |x_v|^k \left(\frac{P_{v-1}}{p_v} \right)^k q_v |\varepsilon_{v+1}|^k \right) \right\} \left\{ \sum_{v=0}^{n-1} q_v \right\}^{k-1}} \\
 &= \boxed{\sum_{n=1}^{\infty} \delta \left(\frac{Q_n}{q_n} \right)^k \left\{ \left(\frac{q_n Q_{n-1}^{k-1}}{Q_n Q_{n-1}} \sum_{v=0}^{n-1} |x_v|^k \left(\frac{P_{v-1}}{p_v} \right)^k q_v |\varepsilon_{v+1}|^k \right) \right\}} \\
 &= \boxed{\sum_{n=1}^{\infty} \delta \left(\frac{Q_n}{q_n} \right)^k \left\{ \left(\frac{q_n}{Q_n Q_{n-1}} \sum_{v=0}^{n-1} |x_v|^k \left(\frac{P_{v-1}}{p_v} \right)^k q_v |\varepsilon_{v+1}|^k \right) \right\}} \\
 &= \boxed{\sum_{v=1}^{\infty} |x_v|^k \delta \left(\frac{Q_n}{q_n} \right)^k \left(\frac{P_v}{p_v} \right)^{k-1} \frac{P_v}{p_v} q_v |\varepsilon_{v+1}|^k \left\{ \sum_{n=v+1}^{\infty} \left(\frac{q_n}{Q_n Q_{n-1}} \right) \right\}} \\
 &= \boxed{\sum_{v=1}^{\infty} |x_v|^k \delta \left(\frac{Q_v}{q_v} \right)^k \left(\frac{P_v}{p_v} \right)^{k-1} \frac{P_v q_v}{Q_v p_v} |\varepsilon_{v+1}|^k} \\
 &= \boxed{\sum_{v=1}^{\infty} \delta \left(\frac{Q_v}{q_v} \right)^k \left(\frac{P_v}{p_v} \right)^{k-1} \frac{P_v q_v}{p_v Q_v} |x_v|^k \left\{ \frac{p_{v+1} Q_{v+1}}{P_{v+1} q_{v+1}} \right\} \delta \left\{ \frac{p_{v+1} Q_{v+1}}{P_{v+1} q_{v+1}} \right\}^k} \\
 &\quad \text{(using condition (2.5) of theorem)} \\
 &= \boxed{\sum_{v=1}^{\infty} \delta \left(\frac{Q_v}{q_v} \right)^k \left(\frac{P_v}{p_v} \right)^{k-1} |x_v|^k \delta \left\{ \frac{P_{v+1}}{p_{v+1}} \right\}^k \delta \left\{ \frac{q_{v+1}}{Q_{v+1}} \right\}^k} \\
 &= \boxed{\sum_{v=1}^{\infty} \delta \left(\frac{P_v}{p_v} \right)^k \left(\frac{P_v}{p_v} \right)^{k-1} |x_v|^k} \\
 &= \boxed{1}
 \end{aligned}$$

$$\begin{aligned}
 \text{Lastly, } \sum_{14} &= M \sum_{v=1}^{\infty} \delta \left(\frac{Q_v}{q_v} \right)^k \left(\frac{Q_v}{q_v} \right)^{k-1} \left| \sum_4 \right|^k \\
 &= M \sum_{v=1}^{\infty} \delta \left(\frac{Q_v}{q_v} \right)^k \left(\frac{Q_v}{q_v} \right)^{k-1} \left| \frac{q_v}{Q_v} \frac{P_v}{p_v} x_v \varepsilon_v \right|^k \\
 &= \boxed{\sum_{n=1}^{\infty} \delta \left(\frac{Q_n}{q_n} \right)^k \left(\frac{Q_n}{q_n} \right)^{k-1} \frac{q_n^k}{Q_n^k} \frac{P_n^k}{p_n^k} |x_n|^k |\varepsilon_n|^k} \\
 &= \boxed{\sum_{n=1}^{\infty} \delta \left(\frac{Q_n}{q_n} \right)^k \frac{q_n}{Q_n} \frac{P_n^k}{p_n^k} |x_n|^k \left(\frac{p_n Q_n}{q_n P_n} \right) \delta \left(\frac{P_n q_n}{Q_n p_n} \right)^k} \\
 &= \boxed{\sum_{n=1}^{\infty} \delta \left(\frac{P_n}{p_n} \right)^k \left(\frac{P_n}{p_n} \right)^{k-1} |x_n|^k}
 \end{aligned}$$

$\square \quad 1$

This completes the proof of theorem.

4 -COROLLARIES :

The following corollaries can be derived from the theorem-

Cor.1:

In the special case when $\delta_n = n^\delta$ our theorem reduces to the theorem of SINHA and KUMAR [10].

Cor.2:

If $\delta_n = 0$ for every n , then our theorem reduces to theorem of SHARMA[8].

Cor.3:

For $\delta_n = 0$ and $k=1$ our theorem reduces to the following theorem –

Let $\{p_n\}$ and $\{q_n\}$ be two sequences of positive real numbers such that-

$$p_{n+1} = \square(p_n) \quad (4.1)$$

$$q_n = \square(q_{n+1}) \quad (4.2)$$

$$p_n Q_n = \square(P_n q_n) \quad (4.3)$$

If $\sum a_n$ is summable $\left| \overline{N}, p_n \right|_k$ then $\sum a_n \varepsilon_n$ is summable $\left| \overline{N}, q_n \right|_k$ if –

$$\frac{Q_n}{q_n} \Delta \varepsilon_n = \square \left[\left(\frac{p_n Q_n}{P_n q_n} \right) \right] \quad (4.4)$$

$$\varepsilon_n = \square \left[\left(\frac{p_n Q_n}{P_n q_n} \right) \right] \quad (4.5)$$

Acknowledgement: I am very thankful to Dr. Rajiv Sinha (Associate Professor, S.M.P.G. College ,Chandausi ,U.P., India), whose great inspiration lead me to complete this paper.

REFERENCES

- [1] BOR,H.- On two summability methods ;*Math. Proc.Cambridge Phil.Soc.97,(1985),147-149.*
- [2] BOR,H. and THORPE,B.-On some absolute summability methods;*Analysis 7,(1987),145-152.*
- [3] BORWEIN D. and CASS,F.P.-Strong Nölund summability *Math Zeith,103,(1968) 91-111.*
- [4] FLETT,T.M.-*Proc.London Math. Soc.7,(1957),113-141.*
- [5] LEINDLER,L.-*Acta math. Hungar 64 (1994) 269-281.*
- [6] MAZHAR S.M. – On $\left| C, \beta \right|_k$ summability factors of infinite series , *Extract du Bulletin de l'Academie royale de Belgique 9classe des sciences* *senance du Samedi 6 mars (1971).*
- [7] MOHAPTRA, R.N.-A note on summability factors, *J. Indian Math. Soc. (1967), 213-224 .*
- [8] SHARMA,N. –Some aspect of weighted mean matrices,*Ph.D. Theses,Kurushetra University (2000).*
- [9] SINHA,P. and KUMAR,H.– A note on $\left| \overline{N}, p_n ; \delta \right|_k$ summability factors of infinite series factors of infinite series , *Tamkang J.of Mathematics , vol.39, Number 3,193-198 Autumn 2008.*
- [10] _____-International Journal of Math., (Communicated).
- [11] SULAIMAN, W.T.-Inclusion theorem for absolute matrix summability method of infinite series ; *Int. Math. Forum,4,no.24 (2009),1181-1189.*

